

Arithmetic properties for cubic partition pairs modulo powers of 3

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Abstract. Let $b(n)$ denote the number of cubic partition pairs of n . In this paper, we aim to provide a strategy to obtain arithmetic properties of $b(n)$. This gives affirmative answers to two of Lin's conjectures.

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1. Introduction and main results

A partition of a natural number n is a nonincreasing sequence of positive integers whose sum equals n . Let $p(n)$ be the number of partitions. One of Ramanujan's most beautiful work is the following three congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Motivated by Ramanujan's result, Chan [2] introduced the cubic partition function $a(n)$ with generating function given by

$$\sum_{n \geq 0} a(n)q^n = \frac{1}{(q; q)_\infty (q^2; q^2)_\infty}, \quad |q| < 1, \quad (1.1)$$

where and in the sequel, we use the standard notation

$$(a; q)_\infty = \prod_{n \geq 0} (1 - aq^n).$$

One of the analogous congruences to Ramanujan's result is

$$a(3n+2) \equiv 0 \pmod{3}, \quad (1.2)$$

which can be deduced easily from the following identity obtained by Chan

$$\sum_{n \geq 0} a(3n+2)q^n = 3 \frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty^4 (q^2; q^2)_\infty^4}. \quad (1.3)$$

Later on, many authors studied other Ramanujan-like congruences for $a(n)$. For instance, Chen and Lin [3] found four new congruences modulo 7, while recently the author and Dastidar [4] obtained two congruences modulo 11.

After Chan's work, many authors also investigated analogous partition functions. For instance, in 2011, Zhao and Zhong [8] studied the cubic partition pair function

$b(n)$ given by

$$\sum_{n \geq 0} b(n)q^n = \frac{1}{(q; q)_\infty^2 (q^2; q^2)_\infty^2}. \quad (1.4)$$

They obtained several congruences similar to Ramanujan's result:

$$\begin{aligned} b(5n + 4) &\equiv 0 \pmod{5}, \\ b(7n + i) &\equiv 0 \pmod{7}, \\ b(9n + 7) &\equiv 0 \pmod{9}, \end{aligned}$$

where $i = 2, 3, 4, 6$. For combinatorial proofs of the first two congruences, the reader may refer to Kim [5] and Zhou [9] respectively. In a recent paper, Lin [6] studied the arithmetic properties of $b(n)$ modulo 27. For instance, he showed the following infinite families of congruences

$$b\left(81\left(7^{2\alpha+1}(7n+k) + \frac{7^{2\alpha+2}-1}{12}\right) + 7\right) \equiv 0 \pmod{27}$$

for $\alpha \geq 0$ and $k = 1, 2, 3, 4, 5, 6$. He also conjectured the following three congruences modulo powers of 3.

$$b(81n + 61) \equiv 0 \pmod{243}, \quad (1.5)$$

and

$$\sum_{n \geq 0} b(81n + 7)q^n \equiv 9 \frac{(q^2; q^2)_\infty (q^3; q^3)_\infty^2}{(q^6; q^6)_\infty} \pmod{81}, \quad (1.6)$$

$$\sum_{n \geq 0} b(81n + 34)q^n \equiv 36 \frac{(q; q)_\infty (q^6; q^6)_\infty^2}{(q^3; q^3)_\infty} \pmod{81}. \quad (1.7)$$

In this paper, we will provide a strategy to give affirmative answers to Lin's conjectures. In fact, for (1.5), we have a stronger result

Theorem 1.1. *For any $n \geq 0$,*

$$b(81n + 61) \equiv 0 \pmod{729}. \quad (1.8)$$

We also obtain

Theorem 1.2. *For any $n \geq 0$,*

$$b(243n + 61) \equiv 0 \pmod{2187}. \quad (1.9)$$

For the remaining two congruences, we show

Theorem 1.3. *Eqs. (1.6) and (1.7) hold for any $n \geq 0$.*

2. Preliminary results

We first introduce two Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$. They are defined by

$$\begin{aligned} \varphi(q) &= \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2} = \sum_{n=-\infty}^{\infty} q^{n^2}, \\ \psi(q) &= \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty} = \sum_{n \geq 0} q^{n(n+1)/2}. \end{aligned}$$

Let

$$w(q) = \frac{(q; q)_\infty (q^6; q^6)_\infty^3}{(q^2; q^2)_\infty (q^3; q^3)_\infty^3},$$

In two recent papers, Zhao and Zhong [8], and Baruah and Ojah [1] obtained the 3-dissection of $\varphi(-q)$ and $1/\varphi(-q)$ respectively. Their results are

Lemma 2.1. *It holds that*

$$\varphi(-q) = \varphi(-q^9)(1 - 2qw(q^3)). \quad (2.1)$$

Lemma 2.2. *It holds that*

$$\frac{1}{\varphi(-q)} = \frac{\varphi(-q^9)^3}{\varphi(-q^3)^4} (1 + 2qw(q^3) + (2qw(q^3))^2). \quad (2.2)$$

For convenience, we write $\xi(q) := 2qw(q^3)$. Multiplying (2.1) and (2.2), we have

$$\frac{\varphi(-q^9)^4}{\varphi(-q^3)^4} (1 - (2qw(q^3))^3) = 1.$$

It follows that

$$\xi(q)^3 = 1 - \frac{\varphi(-q^3)^4}{\varphi(-q^9)^4}. \quad (2.3)$$

We also notice that in [7], Shen introduced 24 identities involving Ramanujan's theta functions. His Eq. (3.2) states

Lemma 2.3. *It holds that*

$$\frac{\psi(q^3)^3}{\psi(q)} = \frac{1}{8q} \left(\frac{\varphi(-q^3)^3}{\varphi(-q)} - \frac{\varphi(-q)^3}{\varphi(-q^3)} \right). \quad (2.4)$$

By Lemmas 2.1 and 2.2, we can rewrite it in the following form which will be frequently used in our proof:

$$\begin{aligned} \frac{1}{\psi(q)} &= \frac{1}{8q\psi(q^3)^3} \left(\frac{\varphi(-q^3)^3}{\varphi(-q)} - \frac{\varphi(-q)^3}{\varphi(-q^3)} \right) \\ &= \frac{1}{8q\psi(q^3)^3} \left(\frac{\varphi(-q^9)^3}{\varphi(-q^3)} (1 + \xi(q) + \xi(q)^2) - \frac{\varphi(-q^9)^3}{\varphi(-q^3)} (1 - \xi(q)^3) \right) \\ &= \frac{1}{8q\psi(q^3)^3} \frac{\varphi(-q^9)^3}{\varphi(-q^3)} (4\xi(q) - 2\xi(q)^2 + \xi(q)^3). \end{aligned} \quad (2.5)$$

Given a series $\sum_{n \geq 0} u(n)q^n$, we say it is $(s, t, \lambda; 4)$ -expressible if we can find $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}_{>0}$ with $\beta \equiv \lambda \pmod{4}$, and three finite sequences $(s_1, s_2, \dots, s_n) \in \mathbb{Z}^n$, $(t_1, t_2, \dots, t_n) \in \mathbb{Z}^n$, $(v_1, v_2, \dots, v_n) := (v(s_1, t_1), v(s_2, t_2), \dots, v(s_n, t_n)) \in \mathbb{Q}^n$ with $s_1 \equiv s_2 \equiv \dots \equiv s_n \equiv s \pmod{4}$, $t_1 \equiv t_2 \equiv \dots \equiv t_n \equiv t \pmod{4}$, and $s_1 + t_1 = s_2 + t_2 = \dots = s_n + t_n$, such that

$$\sum_{n \geq 0} u(n)q^n = \frac{q^\alpha}{\psi(q)^\beta} \sum_{i=1}^n v(s_i, t_i) \varphi(-q)^{s_i} \varphi(-q^3)^{t_i}.$$

The 3-dissection of a $(s, t, \lambda; 4)$ -expressible series has the following property:

Lemma 2.4. *If s , t , and λ satisfy one of the following conditions:*

- (1) $\lambda = 0$, and $s \equiv t \pmod{4}$;
- (2) $\lambda = 2$, and $s \equiv t + 2 \pmod{4}$,

then given an $(s, t, \lambda; 4)$ -expressible series $\sum_{n \geq 0} u(n)q^n$, we can find an integer $\ell \in \{0, 1, 2\}$ such that $\sum_{n \geq 0} u(3n + \ell)q^n$ is also $(s, t, \lambda; 4)$ -expressible.

Proof. If $\sum_{n \geq 0} u(n)q^n$ is $(a, b, \lambda; 4)$ -expressible, we can write it as

$$\sum_{n \geq 0} u(n)q^n = \frac{q^\alpha}{\psi(q)^\beta} \sum_{i=1}^n v(s_i, t_i) \varphi(-q)^{s_i} \varphi(-q^3)^{t_i}.$$

By (2.5), we have

$$\sum_{n \geq 0} u(n)q^n = \frac{q^{\alpha-\beta}}{8^\beta \psi(q^3)^{3\beta}} \sum_{n \geq 0} g(n)q^n,$$

where

$$\sum_{n \geq 0} g(n)q^n = \left(\frac{\varphi(-q^9)^3}{\varphi(-q^3)} \right)^\beta (4\xi(q) - 2\xi(q)^2 + \xi(q)^3)^\beta \sum_{i=1}^n v(s_i, t_i) \varphi(-q)^{s_i} \varphi(-q^3)^{t_i}.$$

Suppose that $\alpha - \beta \equiv \ell \pmod{3}$, we have

$$\sum_{n \geq 0} u(3n + \ell)q^n = \frac{q^{(\alpha-\beta-\ell)/3}}{8^\beta \psi(q)^{3\beta}} \sum_{n \geq 0} g(3n)q^n.$$

Here note that if $\beta \equiv \lambda = 0$ or $2 \pmod{4}$, we have $3\beta \equiv \lambda \pmod{4}$.

Next for any $i = 1, \dots, n$, we aim to obtain the 3-dissection of

$$\sum_{n \geq 0} g_i(n)q^n = \varphi(-q)^{s_i} \varphi(-q^3)^{t_i} \left(\frac{\varphi(-q^9)^3}{\varphi(-q^3)} \right)^\beta (4\xi(q) - 2\xi(q)^2 + \xi(q)^3)^\beta.$$

If $s_i \geq 0$, then by Lemma 2.1,

$$\begin{aligned} & \varphi(-q)^{s_i} \varphi(-q^3)^{t_i} \left(\frac{\varphi(-q^9)^3}{\varphi(-q^3)} \right)^\beta (4\xi(q) - 2\xi(q)^2 + \xi(q)^3)^\beta \\ &= \varphi(-q^3)^{t_i-\beta} \varphi(-q^9)^{s_i+3\beta} (1 - \xi(q))^{s_i} (4\xi(q) - 2\xi(q)^2 + \xi(q)^3)^\beta. \end{aligned}$$

Now we pick up terms of the form q^{3n} and obtain

$$\varphi(-q^3)^{t_i-\beta} \varphi(-q^9)^{s_i+3\beta} \sum_{j=0}^{\lfloor \frac{s_i+3\beta}{3} \rfloor} m_{3j} \xi(q)^{3j}$$

for some m_{3j} . By (2.3), we therefore have

$$\begin{aligned} \sum_{n \geq 0} g_i(3n)q^n &= \varphi(-q)^{t_i-\beta} \varphi(-q^3)^{s_i+3\beta} \sum_{j=0}^{\lfloor \frac{s_i+3\beta}{3} \rfloor} m_{3j} \left(1 - \frac{\varphi(-q)^4}{\varphi(-q^3)^4} \right)^j \\ &= \sum_{j=0}^{\lfloor \frac{s_i+3\beta}{3} \rfloor} v'(s'_{i,j}, t'_{i,j}) \varphi(-q)^{s'_{i,j}} \varphi(-q^3)^{t'_{i,j}}, \end{aligned}$$

where

$$\begin{cases} s'_{i,j} = t_i - \beta + 4j, \\ t'_{i,j} = s_i + 3\beta - 4j, \end{cases}$$

for $j = 0, \dots, \lfloor (s_i + 3\beta)/3 \rfloor$. For both conditions (i) and (ii), it is easy to see that $s'_{i,j} \equiv s_i \pmod{4}$ and $t'_{i,j} \equiv t_i \pmod{4}$ for all j . Similarly, if $s_j < 0$, we can write $\sum_{n \geq 0} g_i(3n)q^n$ as a linear combination of $\varphi(-q)^{s'_{i,j}} \varphi(-q^3)^{t'_{i,j}}$ where

$$\begin{cases} s'_{i,j} = t_i + 4s_i - \beta + 4j, \\ t'_{i,j} = -3s_i + 3\beta - 4j, \end{cases}$$

for $j = 0, \dots, \lfloor (-2s_i + 3\beta)/3 \rfloor$. For both conditions (i) and (ii), we also see that $s'_{i,j} \equiv s_i \pmod{4}$ and $t'_{i,j} \equiv t_i \pmod{4}$ for all j . Furthermore, we have $s'_{i,j} + t'_{i,j} = s_i + t_i + 2\beta$ for all j whenever $s_i \geq 0$ or $s_i < 0$.

Since the above argument holds for all $i = 1, \dots, n$, we therefore can find three new finite sequences $(s'_1, s'_2, \dots, s'_{n'}) \in \mathbb{Z}^{n'}$, $(t'_1, t'_2, \dots, t'_{n'}) \in \mathbb{Z}^{n'}$, $(v'_1, v'_2, \dots, v'_{n'}) := (v'(s'_1, t'_1), v'(s'_2, t'_2), \dots, v'(s'_{n'}, t'_{n'})) \in \mathbb{Q}^{n'}$ with $s'_1 \equiv s'_2 \equiv \dots \equiv s'_{n'} \equiv s \pmod{4}$, $t'_1 \equiv t'_2 \equiv \dots \equiv t'_{n'} \equiv t \pmod{4}$, and $s'_1 + t'_1 = s'_2 + t'_2 = \dots = s'_{n'} + t'_{n'}$, such that

$$\sum_{n \geq 0} u(3n + \ell)q^n = \frac{q^{(\alpha - \beta - \ell)/3}}{\psi(q)^{3\beta}} \sum_{i=1}^{n'} v'(s'_i, t'_i) \varphi(-q)^{s'_i} \varphi(-q^3)^{t'_i}.$$

□

Remark 2.1. The aim of this lemma is to show that if a series $\sum_{n \geq 0} u(n)q^n$ is $(s, t, \lambda; 4)$ -expressible satisfying conditions (i) or (ii), then we may write

$$\sum_{n \geq 0} u(3^k n + \ell)q^n$$

(for some computable ℓ) as the form

$$\frac{q^\alpha}{\psi(q)^\beta} \sum_{i=1}^n v(s_i, t_i) \varphi(-q)^{s_i} \varphi(-q^3)^{t_i}.$$

In this sense, we can study the arithmetic property of $u(3^k n + \ell)$ by studying the coefficients $v(s_i, t_i)$.

3. Proofs

In view of (1.4), we can rewrite the generating function of $b(n)$ as

$$\sum_{n \geq 0} b(n)q^n = \frac{1}{\psi(q)^2 \varphi(-q)^2} = \frac{\varphi(-q^3)^0}{\psi(q)^2 \varphi(-q)^2}. \quad (3.1)$$

Hence $\sum_{n \geq 0} b(n)q^n$ is $(2, 0, 2; 4)$ -expressible. According to Lemma 2.4, we can see that for any positive integer α , we can find some ℓ , such that $\sum_{n \geq 0} b(3^\alpha n + \ell)q^n$ has the form

$$\frac{q^\alpha}{\psi(q)^\beta} \sum_{i=1}^n v(s_i, t_i) \varphi(-q)^{s_i} \varphi(-q^3)^{t_i},$$

where $s_1 \equiv s_2 \equiv \dots \equiv s_n \equiv 2 \pmod{4}$, $t_1 \equiv t_2 \equiv \dots \equiv t_n \equiv 0 \pmod{4}$, and $s_1 + t_1 = s_2 + t_2 = \dots = s_n + t_n$.

Using the method presented in the proof of Lemma 2.4, we first give the detail of the 3-dissection of $\sum_{n \geq 0} b(n)q^n$.

$$\frac{1}{\psi(q)^2 \varphi(-q)^2} = \frac{q^{-2}}{2^6 \psi(q^3)^6} \left(\frac{\varphi(-q^9)^3}{\varphi(-q^3)} \right)^2 (4\xi(q) - 2\xi(q)^2 + \xi(q)^3)^2$$

$$\begin{aligned}
& \times \left(\frac{\varphi(-q^9)^3}{\varphi(-q^3)^4} \right)^2 (1 + \xi(q) + \xi(q)^2)^2 \\
& = \frac{q^{-3} \cdot q}{2^6 \psi(q^3)^6} \frac{\varphi(-q^9)^{12}}{\varphi(-q^3)^{10}} (4\xi(q) - 2\xi(q)^2 + \xi(q)^3)^2 (1 + \xi(q) + \xi(q)^2)^2.
\end{aligned}$$

Extracting terms involving q^{3n+1} and replacing q^{3n} by q^n , we have

$$\sum_{n \geq 0} b(3n+1)q^n = \frac{q^{-1}}{2^6 \psi(q)^6} \left(2\varphi(-q)^2 + 7 \frac{\varphi(-q^3)^4}{\varphi(-q)^2} - 36 \frac{\varphi(-q^3)^8}{\varphi(-q)^6} + 27 \frac{\varphi(-q^3)^{12}}{\varphi(-q)^{10}} \right).$$

Using the same method, we can obtain

$$\begin{aligned}
& \sum_{n \geq 0} b(9n+7)q^n \\
& = \frac{q^{-3}}{2^{24} \psi(q)^{18}} \left(252 \frac{\varphi(-q)^{18}}{\varphi(-q^3)^4} + 16254 \varphi(-q)^{14} + 54054 \varphi(-q)^{10} \varphi(-q^3)^4 \right. \\
& \quad + 54180 \varphi(-q)^6 \varphi(-q^3)^8 - 3679992 \varphi(-q)^2 \varphi(-q^3)^{12} + 33485805 \frac{\varphi(-q^3)^{16}}{\varphi(-q)^2} \\
& \quad - 201778452 \frac{\varphi(-q^3)^{20}}{\varphi(-q)^6} + 846955116 \frac{\varphi(-q^3)^{24}}{\varphi(-q)^{10}} - 2445337188 \frac{\varphi(-q^3)^{28}}{\varphi(-q)^{14}} \\
& \quad + 4746831012 \frac{\varphi(-q^3)^{32}}{\varphi(-q)^{18}} - 6004220418 \frac{\varphi(-q^3)^{36}}{\varphi(-q)^{22}} + 4706441496 \frac{\varphi(-q^3)^{40}}{\varphi(-q)^{26}} \\
& \quad \left. - 2066242608 \frac{\varphi(-q^3)^{44}}{\varphi(-q)^{30}} + 387420489 \frac{\varphi(-q^3)^{48}}{\varphi(-q)^{34}} \right).
\end{aligned}$$

Checking the coefficients 252, 16254, ..., we can see that they are all divisible by 9. It therefore follows that

$$b(9n+7) \equiv 0 \pmod{9}$$

for all $n \geq 0$, which was obtained previously by Zhao and Zhong [8].

For the following dissections, the coefficients become very large. Hence we only display them under modulus $2187 = 3^7$. We have

$$\begin{aligned}
& \sum_{n \geq 0} b(27n+7)q^n \\
& \equiv \frac{q^{-7}}{2^{78} \psi(q)^{54}} \left(252 \frac{\varphi(-q)^{74}}{\varphi(-q^3)^{24}} + 504 \frac{\varphi(-q)^{70}}{\varphi(-q^3)^{20}} + 918 \frac{\varphi(-q)^{66}}{\varphi(-q^3)^{16}} + 2034 \frac{\varphi(-q)^{62}}{\varphi(-q^3)^{12}} \right. \\
& \quad + 396 \frac{\varphi(-q)^{58}}{\varphi(-q^3)^8} + 1755 \frac{\varphi(-q)^{54}}{\varphi(-q^3)^4} + 225 \varphi(-q)^{50} + 1530 \varphi(-q)^{46} \varphi(-q^3)^4 \\
& \quad \left. + 1701 \varphi(-q)^{42} \varphi(-q^3)^8 + 1620 \varphi(-q)^{38} \varphi(-q^3)^{12} \right) \pmod{2187}.
\end{aligned}$$

Continuing the dissection, we have

$$\begin{aligned}
& \sum_{n \geq 0} b(81n+61)q^n \\
& \equiv \frac{q^{-21}}{2^{240} \psi(q)^{162}} \left(729 \frac{\varphi(-q)^{230}}{\varphi(-q^3)^{72}} + 1458 \frac{\varphi(-q)^{226}}{\varphi(-q^3)^{68}} + 1458 \frac{\varphi(-q)^{218}}{\varphi(-q^3)^{60}} + 729 \frac{\varphi(-q)^{214}}{\varphi(-q^3)^{56}} \right.
\end{aligned}$$

$$\begin{aligned}
& + 729 \frac{\varphi(-q)^{194}}{\varphi(-q^3)^{36}} + 1458 \frac{\varphi(-q)^{190}}{\varphi(-q^3)^{32}} + 1458 \frac{\varphi(-q)^{182}}{\varphi(-q^3)^{24}} + 729 \frac{\varphi(-q)^{178}}{\varphi(-q^3)^{20}} \\
& + 729 \varphi(-q)^{158} + 1458 \varphi(-q)^{154} \varphi(-q^3)^4 + 1458 \varphi(-q)^{146} \varphi(-q^3)^{12} \\
& + 729 \varphi(-q)^{142} \varphi(-q^3)^{16} \Big) \pmod{2187}.
\end{aligned}$$

Now by checking the coefficients modulo 729, we can see that

$$b(81n + 61) \equiv 0 \pmod{729},$$

and thus complete the proof of Theorem 1.1.

Furthermore, we can rewrite $\sum_{n \geq 0} b(81n + 61)q^n$ as

$$\begin{aligned}
& \sum_{n \geq 0} b(81n + 61)q^n \\
& \equiv \frac{729q^{-21}}{2^{240}\psi(q)^{162}} \varphi(-q)^{142} \varphi(-q^3)^{16} \\
& \quad \times (1 + \kappa(q)^9 + \kappa(q)^{18})(1 + 2\kappa(q) + 2\kappa(q)^3 + \kappa(q)^4) \pmod{2187},
\end{aligned}$$

where

$$\kappa(q) = \frac{\varphi(-q)^4}{\varphi(-q^3)^4}.$$

According to (2.3), we have

$$\kappa(q) = 1 - 8qw(q)^3,$$

and thus

$$\kappa(q) \equiv 1 - 2qw(q^3) = 1 - \xi(q) \pmod{3}.$$

To prove Theorem 1.2, we note that

$$\begin{aligned}
& \frac{q^{-21}}{2^{240}\psi(q)^{162}} \varphi(-q)^{142} \varphi(-q^3)^{16} (1 + \kappa(q)^9 + \kappa(q)^{18})(1 + 2\kappa(q) + 2\kappa(q)^3 + \kappa(q)^4) \\
& \equiv \frac{(q^3)^{-7}}{\psi(q^3)^{54}} \varphi(-q) \varphi(-q^3)^{47} \varphi(-q^3)^{16} (1 + \kappa(q^3)^3 + \kappa(q^3)^6)(1 - \kappa(q))^4 \\
& \equiv \frac{(q^3)^{-7}}{\psi(q^3)^{54}} \varphi(-q^3)^{63} \varphi(-q^9) (1 + \kappa(q^3)^3 + \kappa(q^3)^6)(1 - \xi(q)) \xi(q)^4 \pmod{3}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{n \geq 0} b(81n + 61)q^n \\
& \equiv \frac{729(q^3)^{-7}}{\psi(q^3)^{54}} \varphi(-q^3)^{63} \varphi(-q^9) (1 + \kappa(q^3)^3 + \kappa(q^3)^6)(1 - \xi(q)) \xi(q)^4 \pmod{2187}.
\end{aligned}$$

Since $\xi(q) = 2qw(q^3)$, we can see that the right hand side contains no term of the form q^{3n} . Hence

$$b(243n + 61) \equiv 0 \pmod{2187}$$

holds for all $n \geq 0$.

We next prove Theorem 1.3. If we extract terms of the form q^{3n} from $\sum_{n \geq 0} b(27n + 7)q^n$ and replace q^3 by q , we have

$$\sum_{n \geq 0} b(81n + 7)q^n$$

$$\begin{aligned}
&\equiv \frac{9q^{-20}w(q)\varphi(-q)^{154}\varphi(-q^3)^4}{2^{239}\psi(q)^{162}} \\
&\quad \times (5 + 8\kappa(q) + 5\kappa(q)^2 + 6\kappa(q)^3 + 6\kappa(q)^4 + 6\kappa(q)^5 + 3\kappa(q)^6 \\
&\quad + 3\kappa(q)^7 + 3\kappa(q)^8 + 8\kappa(q)^9 + 2\kappa(q)^{10} + 8\kappa(q)^{11} + 3\kappa(q)^{12} \\
&\quad + 3\kappa(q)^{13} + 3\kappa(q)^{14} + 6\kappa(q)^{15} + 6\kappa(q)^{16} + 6\kappa(q)^{17} + 5\kappa(q)^{18} \\
&\quad + 8\kappa(q)^{19} + 5\kappa(q)^{20}) \pmod{81}.
\end{aligned}$$

Note that

$$\begin{aligned}
(1-x)^{20} &\equiv 1 + 7x + x^2 + 3x^3 + 3x^4 + 3x^5 + 6x^6 + 6x^7 + 6x^8 \\
&\quad + 7x^9 + 4x^{10} + 7x^{11} + 6x^{12} + 6x^{13} + 6x^{14} + 3x^{15} + 3x^{16} \\
&\quad + 3x^{17} + x^{18} + 7x^{19} + x^{20} \pmod{9}.
\end{aligned}$$

We therefore have

$$\begin{aligned}
&\sum_{n \geq 0} b(81n+7)q^n \\
&\equiv 9 \times \frac{5q^{-20}w(q)\varphi(-q)^{154}\varphi(-q^3)^4}{2^{239}\psi(q)^{162}} (1 - \kappa(q))^{20} \\
&\equiv 9 \frac{(q; q)_{\infty}^{531} (q^6; q^6)_{\infty}^{179}}{(q^2; q^2)_{\infty}^{539} (q^3; q^3)_{\infty}^{175}} \pmod{81}.
\end{aligned}$$

One readily sees from the binomial theorem that

$$(q; q)_{\infty}^9 \equiv (q^3; q^3)_{\infty}^3 \pmod{9}.$$

We therefore conclude that

$$\begin{aligned}
&\sum_{n \geq 0} b(81n+7)q^n \\
&\equiv 9(q^3; q^3)_{\infty}^2 \left(\frac{(q; q)_{\infty}^9}{(q^3; q^3)_{\infty}^3} \right)^{59} \frac{(q^2; q^2)_{\infty}}{(q^6; q^6)_{\infty}} \left(\frac{(q^6; q^6)_{\infty}^3}{(q^2; q^2)_{\infty}^9} \right)^{60} \\
&\equiv 9 \frac{(q^2; q^2)_{\infty} (q^3; q^3)_{\infty}^2}{(q^6; q^6)_{\infty}} \pmod{81}.
\end{aligned}$$

This proves (1.6).

At last, we extract terms of the form q^{3n+1} from $\sum_{n \geq 0} b(27n+7)q^n$ and replace q^3 by q . Then

$$\begin{aligned}
&\sum_{n \geq 0} b(81n+34)q^n \\
&\equiv 9 \frac{q^{-20}w(q)^2\varphi(-q)^{154}\varphi(-q^3)^4}{2^{238}\psi(q)^{162}} \\
&\quad \times (1 + 7\kappa(q) + \kappa(q)^2 + 3\kappa(q)^3 + 3\kappa(q)^4 + 3\kappa(q)^5 + 6\kappa(q)^6 \\
&\quad + 6\kappa(q)^7 + 6\kappa(q)^8 + 7\kappa(q)^9 + 4\kappa(q)^{10} + 7\kappa(q)^{11} + 6\kappa(q)^{12} \\
&\quad + 6\kappa(q)^{13} + 6\kappa(q)^{14} + 3\kappa(q)^{15} + 3\kappa(q)^{16} + 3\kappa(q)^{17} + \kappa(q)^{18} \\
&\quad + 7\kappa(q)^{19} + \kappa(q)^{20})
\end{aligned}$$

$$\begin{aligned}
&\equiv 9 \frac{q^{-20} w(q)^2 \varphi(-q)^{154} \varphi(-q^3)^4}{2^{238} \psi(q)^{162}} (1 - \kappa(q))^{20} \\
&\equiv 36 \frac{(q; q)_\infty^{532} (q^6; q^6)_\infty^{182}}{(q^2; q^2)_\infty^{540} (q^3; q^3)_\infty^{178}} \\
&\equiv 36 \frac{(q; q)_\infty}{(q^3; q^3)_\infty} \left(\frac{(q; q)_\infty^9}{(q^3; q^3)_\infty^3} \right)^{59} (q^6; q^6)_\infty^2 \left(\frac{(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty^9} \right)^{60} \\
&\equiv 36 \frac{(q; q)_\infty (q^6; q^6)_\infty^2}{(q^3; q^3)_\infty} \pmod{81}.
\end{aligned}$$

We therefore end the proof of (1.7).

4. Concluding remarks

Recall that from Lemma 2.4, for any positive integer α , we can find some ℓ , such that $\sum_{n \geq 0} b(3^\alpha n + \ell) q^n$ has the form

$$\frac{q^\alpha}{\psi(q)^\beta} \sum_{i=1}^n v(s_i, t_i) \varphi(-q)^{s_i} \varphi(-q^3)^{t_i},$$

where $s_1 \equiv s_2 \equiv \cdots \equiv s_n \equiv 2 \pmod{4}$, $t_1 \equiv t_2 \equiv \cdots \equiv t_n \equiv 0 \pmod{4}$, and $s_1 + t_1 = s_2 + t_2 = \cdots = s_n + t_n$. In fact, it is easy to show by induction that

$$\ell = \begin{cases} 1 + \frac{3^{\alpha+1}-3}{4} & \alpha \text{ even,} \\ 1 + \frac{3^\alpha-3}{4} & \alpha \text{ odd.} \end{cases}$$

From Theorem 1.2, we see that for $\alpha = 6$ and 7 (and thus $\ell = 547$ in both cases),

$$b(729n + 547) \equiv 0 \pmod{2187}$$

and

$$b(2187n + 547) \equiv 0 \pmod{2187}.$$

However the modulus $2187 = 3^7$ is the best choice since

$$\begin{aligned}
b(547) &= 2135474526556068875092854278074796547960 \\
&= 2^3 \times 3^7 \times 5 \times 41 \times 61 \times 151 \times 11909 \times 5427748132276664632973303.
\end{aligned}$$

We therefore want to know if there is a general family of congruences modulo higher power of 3 for $b(3^\alpha n + \ell)$?

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